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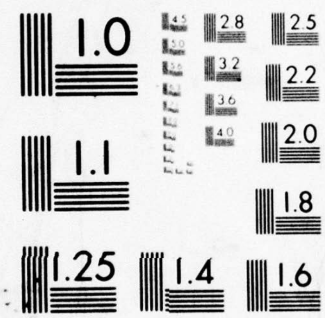

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IN

ECONOMIC SYSTEMS

THE GENESIS OF DYNAMIC SYSTEMS

GOVERNED BY METZLER MATRICES

by

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THE GENESIS OF DYNAMIC SYSTEMS GOVERNED BY METZLER MATRICES

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0. Introduction.

The literature on dynamic systems in economics is vast, and an important part of that deals with systems of differential or of difference equations where the Jacobian of the right-hand side is a Metzler matrix, i.e., a matrix whose off-diagonal elements are non-negative. Such matrices have a wide range of applicability in dynamic economic models, in input-output analysis, in stability analysis of systems of excess demands governing price changes, and in multi-sector and multi-national Keynesian income determination models. Oskar Morgenstern early perceived the importance of such models and encouraged research in them, as seen in the papers by Y.K. Wong and M.A. Woodbury in Morgenstern (3). For a later survey, see (2).

The bulk of this work, as indeed the bulk of the work on dynamic systems in general, concerns what might, in theological terms, be called the eschatology of the system, the questions of the end or final state of the system. In this paper, I want to concentrate on the behavior of the system in its initial phases, its genesis. The problems revolve mainly about the presence of off-diagonal zeroes in the matrix governing the system and of zeroes in some components of the initial conditions and of the forcing terms. If the matrix were strictly positive, for example, then any initial impulse anywhere immediately (or with a lag of one time period in a difference equation system) produces a positive response everywhere. But if there are some zeroes in the matrix, then the transmission of the impulses is delayed. In the case of differential equation systems, the effect appears as a lower rate of growth of the component; instead of increasing linearly, it may increase quadratically or even with a higher power from its initial value of zero.

The discussion of this subject involves what appears to be a new concept, that of first-positivity. A sequence is first-positive if the first non-zero element is positive. Similar definitions can be given for sequences of vectors and matrices. The methods bring together a number of elementary concepts from diverse fields, including matrix algebra, the theory of relations, and the theory of differential equations.

In section 1, the concept of first-positivity is introduced, and some elementary properties developed. In section 2, there is a digression on some properties of relations which will be useful in the sequel. Section 3 studies first-nonzero and connectivity properties of the sequence formed by the powers of a matrix; at this stage, there is no restriction to Metzler matrices. Section 4 discusses a property of matrix exponentials, which are used in expressing the solution to a system of differential equations. In section 5, we draw the previous results together in application to the first-positivity and connectivity properties of the sequence of powers of a Metzler matrix.

The results to which the earlier sections were leading are contained in section 6 and section 7. These characterize the geneses of systems of difference equations and of differential equations, respectively. In the first, the time of first-positivity of a specific component is expressed in terms of the connectivity properties of the matrix, the specification of the positive components of the starting values, and the first-positivity properties of the forcing function. In the second, the order of increase of a specific component is expressed in terms of the same factors.

1. First-positivity.

In the following, we use "D" as an abbreviation for "Definition." In general, small letters, such as "x," stand for sequences (functions on the non-negative integers) of scalars or vectors; a capital letter, such as "A," stands for a sequence of matrices; "$x(n)$" or "$A(n)$" will be the value of x or A, respectively, at integer n; "x_α" or "$A_{\alpha\beta}$" will be the scalar sequence formed by considering only the component α of the vector or the element (α,β) of the matrix sequence.

D.1. For a scalar sequence x, define $\nu(x) = \min \{ n \mid x(n) \neq 0 \}$. We refer to $\nu(x)$ as the non-zero index of x.

Note that the non-zero index is not necessarily defined, since x might be identically zero. This problem of lack of definition occurs persistently but can be dealt with, as will be noted later.

D. 2. $x^\nu \triangleq x[\nu(x)]$

Thus x is the value of the first element in the sequence x which is not zero. The symbol, "$\triangleq$," is read, "equals by definition."

D. 3. (x is first-positive) $\triangleq (x^\nu > 0)$.

Now let x be a sequence of vectors. Each of the above definitions is still allowed to hold, but must be applied to each component. That is, for each component α of the vector, x_α is a scalar sequence. Then $\nu(x_\alpha)$ is defined by D. 1, and similarly, $(x_\alpha)^\nu$ by D. 2. It will be useful to interpret x to be a function from a finite domain (the domain of its components) to the space of scalar sequences. Let F be this domain. Then $\nu(x)$ will be interpreted as a function, the value for α being $\nu(x_\alpha)$. It must be emphasized, however, that $\nu(x)$ might not be defined for all elements of F, for it can happen that the sequence x_α is identically zero for some α, and therefore $\nu(x)(\alpha) = \nu(x_\alpha)$ is not defined for that value of α.

D. 4. If x is a sequence of vectors, the function $\nu(x)$ is defined by the relation, $\nu(x)(\alpha) = \nu(x_\alpha)$.

D. 5. $x^\nu(\alpha) = x_\alpha[\nu(x_\alpha)] = x_\alpha{}^\nu$,

so that the function x^ν has the same domain of definition as $\nu(x)$.

The analogue of D.3 for vectors is a vector inequality, i.e., we regard a sequence of vectors as first-positive if each component is. However, we have to account for the possibility that the domain of definition of x might not be the entire possible domain F. We need a convention for the meaning of an expression like $f(\alpha) > 0$, or, more generally, for an expression, $f(\alpha) > g(\alpha)$, where one or both of the functions f and g may be undefined at a particular point α. We shall regard the inequality as holding if f is undefined at α and g is defined there or if both are defined and $f(\alpha)$ is indeed bigger than $g(\alpha)$, and not otherwise.

Convention 1. If the functions f and g are both defined at α, then the expression, $f(\alpha) > g(\alpha)$, has its usual meaning; otherwise, it holds if and only if f is not defined at α and g is.

A similar convention will hold for equality.

Convention 2. If the functions f and g are both define at α, then the expression, $f(\alpha) = g(\alpha)$, has its usual meaning; otherwise, it holds if and only if neither f nor g are defined at α.

If we use the usual symbol,

D.6. $\text{dom } f = \{\alpha \mid f(\alpha) \text{ is defined}\}$,

then Convention 1 says that $f(\alpha) > g(\alpha)$ if $\alpha \in \text{dom } g$ and $\alpha \notin \text{dom } f$, while Convention 2 implies that $f(\alpha) = g(\alpha)$ if $\alpha \notin \text{dom } f$, $\alpha \notin \text{dom } g$.

Inequality among functions has the usual meaning that the inequality holds for all values of the argument, with, however, the conventions above observed. From the above remarks, it is easy to note that,

$f \leq g$ implies $\text{dom } f \subset \text{dom } g$.

With these conventions, the definition of first-positivity, D. 3, remains valid for sequences of vectors. It means that each component is first-positive if not identically zero.

The following simple lemma holds for adding first-positive sequences, whether of vectors or of scalars.

Lemma 1. If x^i is first-positive for each i, then $\sum_i x^i$ is first-positive, and $\nu(\sum_i x^i) = \min_i \nu(x^i)$.

Proof: Let $\bar{\nu} = \min_i \nu(x^i)$, $P = \{ i \mid \nu(x^i) = \bar{\nu} \}$. If $n < \bar{\nu}$, then $n < \nu(x^i)$ for all i, so that $x^i(n) = 0$ for all i, by D.1, and therefore $\sum_i x^i(n) = 0$. On the other hand, $x^i(\bar{\nu}) = x^{i\nu} > 0$ for $i \in P$, by D.2 and then D.3, $\nu(x^i) > \bar{\nu}$ for $i \notin P$, and therefore $x^i(\bar{\nu}) = 0$ for $i \notin P$. Hence, $\sum_i x^i(\bar{\nu}) > 0$, so that the Lemma holds.

Although the proof has been stated for scalars, it holds, with suitable interpretation for vectors; the operator, "$\min_i$," in the statement of the Lemma must be taken to hold component-wise.

Also, it is useful to note how the Conventions are used. The proof as given seems to require that $\nu(x^i)$ is defined for all i. If, however, it is defined only for some i, then $\overline{\nu}$ is taken as the minimum over all i for which it is defined. That indeed is the interpretation implied by Conventions 1 and 2. Then the argument is valid in every detail; in particular, if $\nu(x^i)$ is not defined for some i, then certainly, by Convention 1, $\nu(x^i) > \overline{\nu}$. On the other hand, in just that case, $x^i(n) = 0$ for all n, and therefore certainly, $x^i(\overline{\nu}) = 0$. In the future, all proofs will be carried on as though all functions were defined; the correction for the cases of lack of definition can easily be supplied by the reader.

The same definitions will be needed for sequences of matrices; however, since a matrix can be thought of as a vector, there is no need for additional definitions. (Square) matrices can be thought of as functions on a domain of the form, F x F, there F is finite, and "x" denotes Cartesian product. In this case, if A is a sequence of matrices, $\nu(A)$ is a function of two variables, representing the rows and the columns.

A useful concept in expressing the solution of systems of difference equations is the <u>convolution</u> of two sequences, a term borrowed from probability theory. It is the same as the expression for the distribution of the sum of two independent non-negative random variables.

D. 7. If x and y are two sequences, then the sequence x*y is defined by,

$$(x*y)\ (n) = \sum_{j=0}^{n} x(j)\ y(n-j).$$

First, suppose x and y are scalar sequences. From D.1, $x(j) = 0$ if $j < \nu(x)$, and $y(n-j) = 0$ if $n-j < \nu(y)$, or, equivalently, $j > n-\nu(y)$. Hence,

$$x(j)\ y(n-j) = 0 \text{ unless } \nu(x) \leq j \leq n-\nu(y).$$

If $n < \nu(x) + \nu(y)$, then $\nu(x) > n - \nu(y)$, so that $x(j)\ y(n-j) = 0$ for all j, and therefore $(x*y)\ (n) = 0$. If $n = \nu(x) + \nu(y)$, then $x(j)\ y(n-j) = 0$ except for $j = \nu(x) = n - \nu(y)$, so that $(x*y)\ (n) = x[\nu(x)]\ y[\nu(y)] = x^{\nu}y^{\nu} \neq 0$, by D.2 and D.1. Hence, for scalar sequences, $(x*y)\ (n) = 0$ for $n < \nu(x) + \nu(y)$, $\neq 0$ for $n = \nu(x) + \nu(y)$, so that, by D.1, $\nu(x*y) = \nu(x) + \nu(y)$, and, by D.2, $(x*y)^{\nu} = x^{\nu}y^{\nu}$. If it is also assumed that x and y are first-positive, then, by D.3, $x^{\nu} > 0$, $y^{\nu} > 0$, and therefore $(x*y)^{\nu} > 0$, so that x*y is first-positive.

<u>Lemma 2</u>. If x and y are scalar sequences, then $\nu(x*y) = \nu(x) + \nu(y)$, and $(x*y)^{\nu} = x^{\nu}\ y^{\nu}$. If in addition x and y are first-positive, then so is x*y.

The definition of a convolution can be applied, not only to scalars but also to vectors and pairs consisting of matrices and vectors, with the proper interpretation of multiplication in the definition. Let x and y be vector sequences, each vector being of the same number of components, and let multiplication be interpreted as the taking of an inner product,

$$x(j)\ y(n-j) = \sum_{\beta} x_{\beta}(j)\ y_{\beta}(n-j),$$

so that,

$$(x*y)(n) = \sum_{j=0}^{n} \sum_{\beta} x_\beta(j)\, y_\beta(n-j) = \sum_{\beta} \sum_{j=0}^{n} x_\beta(j)\, y_\beta(n-j) = \sum_{\beta} (x_\beta * y_\beta)(n).$$

Suppose in addition that x and y are first-positive. Then, for each β, x_β and y_β are first-positive, and therefore $x_\beta * y_\beta$ is first-positive, by Lemma 2, and $x*y$ is first-positive by Lemma 1. From Lemmas 1 and 2,

$$\nu(x*y) = \min_{\beta} \nu(x_\beta * y_\beta) = \min_{\beta} [\nu(x)(\beta) + \nu(y)(\beta)]\ . \qquad (2)$$

We will apply this result to multiplication of a matrix sequence A by a sequence of conforming column vectors, x, where both A and x are first-positive. Equation (2) can be applied, with each row of A, in turn, replacing x, and x replacing y. The sequence $A*x$ is a sequence of column vectors.

Lemma 3. Let A be a first-positive sequence of matrices, and x a first-positive sequence of column vectors conforming with A. Then $\nu(A*x)(\alpha) = \min_{\beta} [\nu(A)(\alpha,\beta) + \nu(y)(\beta)]$ and $A*x$ is first-positive.

2. Relations, their Powers, and Chains.

A relation is simply a set of ordered pairs. In the main application in this paper, the relation, C(A), is defined by the condition that $\alpha\ C(A)\ \beta$ is and only if $A_{\alpha\beta} \neq 0$, for a given matrix A; thus, C(A) is the set of all ordered pairs for which this condition holds. Typically, then, a relation, R, is some subset of F x F. In this section, however, the relation R is arbitrary.

A particularly interesting relation is the identity relation, E.

D.8. $(\alpha\ E\ \beta) \overset{\Delta}{=} (\alpha = \beta)$

Following Quine (4), p. 213, the relative product of two relations, R & S, is defined by,

D.9. $(\alpha\ R|S\ \beta) \overset{\Delta}{=}$ (for some γ, $\alpha\ R\ \gamma$ and $\gamma\ S\ \beta$).

Like any other form of multiplication, the relative product can be used to define the powers of a relation inductively.

D.10. $R^0 = E,\ R^{n+1} = R^n | R$.

This definition can be given a useful alternative form by introducing the concept of an R-chain.

D.11. (σ is an R-chain of length n from α to β) $\overset{\Delta}{=}$ (σ is a function defined on the integers $0,\ldots,n$, $\sigma(0) = \alpha$, $\sigma(n) = \beta$, and $\sigma(i-1)\ R\ \sigma(i)$ for $i = 1,\ldots,n$)

In short, an R-chain is an ordered sequence of n+1 elements, such that the relation R holds between every successive pair.

It is intuitively obvious and can easily be proved by induction that the relation R^n holds if and only if there is an R-chain of length n connecting the two elements.

Lemma 4. $\alpha\ R^n\ \beta$ if and only if there exists an R-chain of length n from α to β.

For a given α and β, there may be R-chains of different lengths from one to the other (of course, it is also possible that there are no R-chains of any length from α to β).

D. 12. (σ is a shortest R-chain from α to β) $\stackrel{\Delta}{=}$ (for some n and all m n, σ is an R-chain of length n from α to β and there is no R-chain of length m from α to β).

Shortest chains have a property which will be useful later.

Lemma 5. A shortest R-chain is a one-one function.

Proof: We seek to prove that if σ is a shortest R-chain from α to β and if $\sigma(i_1) = \sigma(i_2)$, then $i_1 = i_2$. Let $\underline{i} = \min(i_1, i_2)$, $h = \max(i_1, i_2) - \underline{i}$. Clearly, $h \geqq 0$ by definition; we seek to prove that $h = 0$, or, equivalently, that $h \subset 0$. To this end, we construct an R-chain of length $n-h$ from α to β, where σ is an R-chain of length n; since σ is a shortest R-chain, it follows by definition that $n-h \geqq n$, or $h \leqq 0$. By assumption,

$$\sigma(\underline{i}) = \sigma(\underline{i} + h). \tag{3}$$

Define a function, σ' on the integers $0,\ldots,n-h$, as follows:

$$\sigma'(i) = \sigma(i),\ 0 \leqq i < \underline{i}, \tag{4a}$$

$$= \sigma(i+h),\ \underline{i} \leqq i \leqq n-h. \tag{4b}$$

If $\underline{i} = 0$, then from (4b) and (3), $\sigma'(0) = \sigma(h) = \sigma(0)$. If $\underline{i} > 0$, then $\sigma'(0) = \sigma(0)$ from (4a), so that $\sigma'(0) = \sigma(0)$ in either case. Also, from (4b), $\sigma'(n-h) = \sigma(n)$. Since σ is an R-chain of length n from α to β, $\sigma(0) = \alpha$ and $\sigma(n) = \beta$; hence, we have shown that $\sigma'(0) = \alpha$ and $\sigma'(n-h) = \beta$.

To show that σ' is an R-chain of length $n-h$ from α to β, it remains, by D.11, to show that $\sigma'(i-1)\ R\ \sigma'(i)$, $1 \leqq i \leqq n-h$. If $i < \underline{i}$, this follows immediately from (4a). If $i > \underline{i}$, then $i-1 \geqq \underline{i}$; by (4b), $\sigma'(i-1) = \sigma(i+h-1)$, $\sigma'(i) = \sigma(i+h)$, and, since σ is an R-chain, $\sigma(i+h-1)\ R\ \sigma(i+h)$, and therefore $\sigma'(i-1)\ R\ \sigma'(i)$.

Finally, let $i = \underline{i}$; then $\underline{i} > 0$. In this case, $\sigma'(i-1) = \sigma(i-1)$, while $\sigma'(i) = \sigma(i+h) = \sigma(\underline{i} + h) = \sigma(\underline{i}) = \sigma(i)$, by (4a), (4b), and (3). Since σ is an R-chain, $\sigma(i-1)\ R\ \sigma(i)$; hence, $\sigma'(i-1)\ R\ \sigma'(i)$. Therefore all the conditions of D.11 are satisfied for σ', so there is an R-chain of length $n-h$ from α to β, and therefore $h \leqq 0$, verifying the lemma.

To any fixed α, there is associated the set $\{\alpha \mid \alpha R \beta\}$; we introduce the notation $R\ \beta$ to stand for this set. More generally, if there is a set of values of β, say S, then $R\ S$, will stand for the union of the sets $R\beta$, for $\beta \in S$.

D.13. $R\ S \stackrel{\Delta}{=} \{\alpha \mid$ for some β, $\alpha\ R\ \beta$ and $\beta \in S\}$.

It is easy to see that,

$$R(S \cup T) = (R\ S) \cup (R\ T), \tag{5}$$

where $A \cup B$ is the union of the sets A and B.

More generally, if S_i is an index set of sets, with i varying over a set i,

$$R \bigcup_i S_i = \bigcup_i R\ S_i\ , \tag{6}$$

where $\bigcup_i S_i$ is the union of the sets S_i.

3. Connectivity of a Matrix and Nonzero Index for the Sequence of Powers of a Matrix

The following connectivity relation can be associated in a natural way with any given matrix A.

D.14. $(\alpha\ C(A)\ \beta\ ) \stackrel{\Delta}{=} (A_{\alpha\beta} \neq 0)$.

The connectivity relation for a power of A will also be of interest.

The sequence of powers of a matrix, A^n, $n \geq 0$, where $A^0 = I$, is a particular sequence, and a nonzero index function can be associated with that sequence. As a matter of notation, we must distinguish between a particular element of the sequence and the name of the sequence. For this purpose, we borrow the functional abstractor notation from mathematical logic (see, e.g. (4), p. 226). In general, for any function which takes on the value f(x) at the point x, we will mean by $\lambda_x\ f(x)$ the name of the function which takes on these values. In this paper, we will apply the notation only to sequences, where the variable is n.

D.15. $\lambda_n\ f(n)$ is the function which takes on the value f(n) when the argument takes on the value n.

Thus, $\lambda_n\ A^n$ is the sequence of powers of A. Associated with this sequence is a nonzero index function, $\nu(\lambda_n\ A^n)$, defined over ordered pairs. By D.1, D.4, and D.14;

$$\nu(\lambda_n\ A^n)\ (\alpha,\ \beta) = \min\{\ n\ |\alpha\ C(A^n)\beta\ \}. \tag{7}$$

Note that,

$$C(A^0) = C(I) = E, \tag{8}$$

so that $\alpha\ C(A^0)\beta$ holds if and only if $\alpha = \beta$; therefore,

$$\nu(\lambda_n\ A^n)(\alpha,\beta) = 0 \text{ if and only if } \alpha = \beta. \tag{9}$$

In view of (7), $\nu\ (\lambda_n\ A^n)$ is defined for a particular pair (α,β) if and only if $\alpha C(A^n)\beta$ for some n. In set-theoretic language, define the relation, $K^\nu\ (A)$, by

D.16. $K^\nu\ (A) = \bigcup_{n=0}^{\infty} C(A^n)$,

where $\bigcup_{n=0}^{\infty}$ is the union of the relations $C(A^n)$; remember that a relatioı is a particular kind of set. Then,

$$K^\nu\ (A) = \text{dom}\ \nu(\lambda_n\ A^n) = \text{dom}\ (\lambda_n\ A^n)^\nu\ . \tag{10}$$

We will explore that effects on the nonzero index and related concepts for a sequence of powers of a matrix of altering the matrix by adding a constant to the diogonal elements. As a preliminary, it is noted that the binomial theorem is valid for pairs of matrices which commute with each other.

$$(A + B)^n = \sum_{r=0}^{n} \binom{n}{r} A^r\ B^{n-r} \text{ if } AB = BA.$$

Let B be a scalar multiple of the identity matrix, B = sI for some scalar s. Then sI commutes with every matrix A. Note that $(sI)^{n-r} = s^{n-r}\ I^{n-r} = s^{n-r}\ I$.

$$(A + sI)^n = \sum_{r=0}^{n} \binom{n}{r} s^{n-r}\ A^r \tag{11}$$

D.17. $(A \equiv B \bmod I) \stackrel{\Delta}{=} (A\text{-}B = sI \text{ for some scalar } s)$

Note that the relation is symmetric. Suppose it holds for two matrices, A and B. From (11),

$$(B^n)_{\alpha\beta} = \sum_{r=0}^{n} \binom{n}{r} s^{n-r}\ (A^r)_{\alpha\beta} \text{ for some scalar s.} \tag{12}$$

Clearly, if $(B^n)_{\alpha\beta} \neq 0$, then it must be that $(A^r)_{\alpha\beta} \neq 0$ for some $r \leq n$. In particular, let $n = \nu(\lambda_n B^n)(\alpha,\beta)$. Then it follows that,

$$\nu(\lambda_n A^n)(\alpha,\beta) \leq \nu(\lambda_n B^n)(\alpha,\beta).$$

But since the relation, $A \equiv B \bmod I$, is symmetric, this must hold with A and B interchanged; also, it holds for any α and β.

$$\nu(\lambda_n A^n) = \nu(\lambda_n B^n).$$

If two functions are equal, they have the same domain of definition, by Convention 2; hence, from (10), $K^\nu(A) = K^\nu(B)$. Further, if we set $n = \nu(\lambda_n A^n)(\alpha,\beta) = \nu(\lambda_n B^n)(\alpha,\beta)$, then, by (7), $(B^r)_{\alpha\beta} = 0$ for $r < n$; from (12), $(B^n)_{\alpha\beta} = (A^n)_{\alpha\beta}$. By D.5,

$$(\lambda_n A^n)^\nu = (\lambda_n B^n)^\nu .$$

Theorem 1. If $A \equiv B \bmod I$, then (a) $\nu(\lambda_n A^n) = \nu(\lambda_n B^n)$; (b) $(\lambda_n A^n)^\nu = (\lambda_n B^n)^\nu$; and (c) $K^\nu(A) = K^\nu(B)$.

The relation C(A) measured what might be termed the direct connectivity of the matrix. Two elements may be indirectly connected through a chain of direct connections. In view of Lemma 4, it is natural to define the connectivity index of a matrix (a function, not a number) as smallest power of C(A) which holds between two elements.

D.18. $\delta(A)(\alpha,\beta) = \min \{ n \mid \alpha[C(A)]^n \beta \}$

Note that, since $[C(A)]^0 = E$, $\delta(A)(\alpha,\beta) = 0$ if and only if $\alpha = \beta$. Also, if $A_{\alpha\beta} \neq 0$ and $\alpha \neq \beta$, then $\alpha[C(A)]^1\beta$ while not $\alpha\, C(A)^0 \beta$, so that $\delta(A)(\alpha,\beta) = 1$. From D.18, the domain of definition of the connectivity index is precisely the set of ordered pairs for which the relation $[C(A)]^n$ holds for some n. Define,

D.19. $K^\delta(A) = \bigcup_{n=0}^{\infty} [C(A)]^n.$

Then,

$$K^\delta(A) = \text{dom}\, \delta(A). \qquad (13)$$

It is convenient to introduce the notation,

$\Sigma(A)(\alpha,\beta, n)$ is the set of C(A)-chains of length n from α to β. (14)

Then, from

$\alpha[C(A)]^n\beta$ if and only if $\Sigma(A)(\alpha,\beta,n)$ is non-empty. (15)

We investigate the effect on the connectivity index of a change in the diagonal elements of the matrix. (Although for our later purposes, only a constant change is relevant, the results hold for any change in the diagonal elements.) Suppose therefore A - B is a diagonal matrix; note again that this relation is symmetric. Let σ be a C(A)-chain of length $\delta(A)(\alpha,\beta)$. By D.18, it is a shortest C(A)-chain from α to β, and therefore σ is a one-one function by Lemma 5. In particular, $\sigma(i-1) \neq \sigma(i)$, $1 \leq i \leq \delta(A)(\alpha,\beta)$. Since A-B is diagonal,

$$A_{\sigma(i-1),\sigma(i)} = B_{\sigma(i-1),\sigma(i)}. \qquad (16)$$

Since σ is a C(A-chain, $\sigma(i-1)\, C(A)\, \sigma(i)$ for all i, or, by D.14, $A_{\sigma(i-1),\sigma(i)} \neq 0$, all i. By (16), $B_{\sigma(i-1),\sigma(i)} \neq 0$ for all i, so that σ is a C(B)-chain. In the notation (14),

$$\Sigma(A)\,(\alpha,\ \beta,\ \ \delta(A)\,(\alpha,\beta)\ ) \subset \Sigma(B)\,(\alpha,\ \beta,\ \delta\,(A)\,(\alpha,\beta)\ ).$$

If $\delta(A)$ is indeed defined at (α,β), then the left-hand set is non-empty and therefore so is the right-hand set. By D.18 and (15),

$$\delta(B)\,(\alpha,\beta) \leqq \delta(A)\,(\alpha,\beta).$$

This statement also holds if $\delta(A)$ is undefined at (α,β) by Conventions 1 and 2. By the symmetry, the inequality holds with A and B interchanged, so that $\delta(A) = \delta(B)$. Since the two functions are equal, they have the same domain of definition. From (13), then, we have

Theorem 2. If $A - B$ is a diagonal matrix, then $\delta(A) = \delta(B)$ and $K^{\delta}(A) = K^{\delta}(B)$.

4. Matrix Exponentials

As is well known, solutions to systems of linear differential equations with constant coefficients can be expressed simply in terms of the exponential of the matrix of coefficients.

D.20. $e^{A} \triangleq \sum_{n=0}^{\infty} A^{n}/n!$

The infinite series converges absolutely for all A, so that e^{A} is defined and the series can be rearranged at will. If we add the scalar multiple of the identity matrix to A, the value of the exponential can be expressed with the aid of (11).

$$e^{A+sI} = \sum_{n=0}^{\infty} (A+sI)^{n}/n! = \sum_{n=0}^{\infty} (1/n!) \sum_{i+j=n} (n!/i!j!)\, s^{i} A^{j}$$

$$= \sum_{n=0}^{\infty} \sum_{i+j=n} (s^{i}/i!)(A^{j}/j!) = \Big(\sum_{i=0}^{\infty} s^{i}/i!\Big)\Big(\sum_{j=0}^{\infty} A^{j}/j!\Big)$$

$$= e^{s} e^{A}$$

Lemma 6. If $A \equiv B \bmod I$, then $e^{B} = p\, e^{A}$ for some positive scalar p.

Corollary 1. If $A \equiv B \bmod I$, then $e^{A} \geqq 0$ if and only if $e^{B} \geqq 0$.

Corollary 2. If $= B \bmod I$, then $C(e^{A}) = C(e^{B})$.

5. First-Positivity and Connectivity of Non-negative and Metzler Matrices.

First, make the obvious observation that if $A \geqq 0$, then $\alpha C(A)\beta$ if and only if $A_{\alpha\beta} > 0$. Recall,

$$(A^{n+1})_{\alpha\beta} = \Sigma_{\gamma}(A^{n})_{\alpha\gamma}\, A_{\gamma\beta}\ .$$

If $A \geqq 0$, then $A^{n} \geqq 0$ for all n. Hence, the right-hand side is a sum of non-negative terms and is positive if and only if at least one is positive. Therefore, $\alpha C(A^{n+1})\ \beta$ if and only if, for some $\delta, (A^{n})_{\alpha\beta}\, A_{\alpha\beta} > 0$, or, equivalently, if and only if, for some γ, $\alpha C(A^{n})_{\gamma}$ and $\gamma C(A)\beta$. In the notation of D.9,

$$C(A^{n+1}) = C(A^{n})\,|\,C(A). \qquad (17)$$

Lemma 7. If $A \geqq 0$, $C(A^{n}) = [C(A)]^{n}$

Proof: For $n = 0$, we know that $C(A^{0}) = E = [C(A)]^{0}$. Suppose the Lemma is true for n. Then, from (17),

$$C(A^{n+1}) = [C(A)]^{n}\,|\,C(A) = [C(A)]^{n+1},$$

by D.10.

From (7) and the definition of $\delta(A)$, D.18, Lemma 7 immediately implies that

$\nu(\lambda_n A^n) = \delta(A)$ when $A \geqq 0$; the two functions have the same domain of definition, so that $K^\nu(A) = K^\delta(A)$ by (10) and (13).

Since $A^n \geqq 0$, it follows immediately from D.5 that $(\lambda_n A^n)^\nu \geqq 0$; but by D.4, it must be that $(\lambda_n A^n)^\nu(\alpha,\beta) \neq 0$ for all (α,β) in the domain of definition, so that $(\lambda_n A^n)^\nu > 0$, that is, the matrix sequence, $\lambda_n A^n$, is first-positive.

Finally, from the definition of an exponential, D.20, $e^A \geqq 0$ when $A \geqq 0$. Since the definining series is a sum of non-negative terms,

$$(e^A)_{\alpha\beta} > 0 \text{ if and only if } (A^n)_{\alpha\beta} > 0 \text{ for some } n,$$

so that,

$$C(e^A) = \bigcup_{n=0}^{\infty} C(A^n) = K^\nu(A),$$

by D.16.

Lemma 8. If $A \geqq 0$, then $\lambda_n A^n$ is first-positive, $\nu(\lambda_n A^n) = \delta(A)$, $e^A \geqq 0$, and $K^\nu(A) = K^\delta(A) = C(e^A)$.

The main mathematical result of the paper is that this Lemma holds not merely for non-negative but for all Metzler matrices. We recall the definition.

D.21. A is Metzler if $A_{\alpha\beta} \geqq 0$ for $\alpha \neq \beta$.

A simple and useful relation between Metzler and non-negative matrices is the following:

$$A \text{ is Metzler if and only if there exists } B \geqq 0 \text{ such that } A \equiv B \bmod I. \tag{18}$$

Theorem 3. If A is Metzler, then $\lambda_n A^n$ is first-positive, $\nu(\lambda_n A^n) = \delta(A)$, $e^A \geqq 0$, and $K^\nu(A) = K^\delta(A) = C(e^A)$.

Proof: Choose B as in (18). Then, from Theorem 1, Lemma 8, and Theorem 2,

$$\nu(\lambda_n A^n)^\nu = \nu(\lambda_n B^n)^\nu = \delta(B) = \delta(A).$$

From Theorem 1 and Lemma 8,

$$(\lambda_n A^n)^\nu = (\lambda_n B^n)^\nu > 0.$$

From Lemma 8, $e^B \geqq 0$, and therefore from Corollary 1, $e^A \geqq 0$. Finally, from Corollary 2, Lemma 8, and Theorem 1, $C(e^A) = C(e^B) = K^\nu(B) = K^\nu(A)$, while from Lemma 8 and Theorem 2, $K^\nu(B) = K^\delta(B) = K^\nu(A)$.

The importance of this theorem is that the qualitative behavior of the powers and the exponential of a Metzler matrix can be inferred solely from its connectivity properties. These depend only on the location of the off-diagonal zeroes and are independent both of the diagonal elements and of the magnitudes of the non-zero off-diagonal elements. Thus, if we raise a Metzler matrix to successively higher powers we know that in each place in the matrix, the first non-zero element (if any) will be positive and the power for which the non-zero entry occurs is equal to the length of the shortest chain from the row element to the column element through non-zero entries.

A side consequence of the analysis is a pair of what are apparently new necessary and sufficient conditions for a matrix to have the Metzler property.

Theorem 4. Each of the following conditions is necessary and sufficient that A be a Metzler matrix: (a) $\lambda_n A^n$ is first-positive; (b) $e^{At} \geqq 0$ for all $t > 0$.

Proof: (a) Necessity has already been shown in Theorem 3. Suppose, then, that the sequence $\lambda_n A^n$ is first-positive. The pairs (α,β) can be classified according as $\nu(\lambda_n A^n)(\alpha,\beta)$ is 0, 1, or greater than 1. In the first case, as remarked in (9), $\alpha = \beta$. In the second, we must have $(A^1)_{\alpha\beta} \neq 0$, by definition, and therefore $A_{\alpha\beta} > 0$, since $\lambda_n A^n$ is first-positive. In the third case, $(A^1)_{\alpha\beta} = 0$ by definition of the non-zero index. Hence, if $\alpha \neq \beta$, $A_{\alpha\beta} \geqq 0$, so that A is a Metzler matrix.

(b) If A is Metzler and $t > 0$, a scalar, then At is also Metzler, and $e^{At} \geqq 0$ by Theorem 3. Conversely, suppose that $e^{At} \geqq 0$ for all $t > 0$. Note that $e^{At} = I$. when $t = 0$, that,

$$\frac{d(e^{At})}{dt} = A\, e^{At},$$

so that,

$$\left.\frac{d(e^{At})}{dt}\right|_{t=0} = A,$$

and that, by definition,

$$\left.\frac{d(e^{At})}{dt}\right|_{t=0} = \lim_{t \to 0} \frac{(e^{At} - I)}{t}.$$

From the hypothesis and (18), $e^{At} - I$ is Metzler for $t > 0$, and therefore $(e^{At} - I)/t$ is a Metzler matrix for $t > 0$. Hence, A is a limit of Metzler matrices; since the set of Metzler matrices is clearly closed, from the definition, A must be a Metzler matrix.

6. Genesis of a System of Difference Equations with Metzler Matrix as Jacobian

We consider the system of difference equations,

$$x(n+1) = Ax(n) + b(n), \tag{19}$$

where we assume that A is a Metzler matrix, b a first-positive sequence, and $x(0) \geqq 0$. To express the solution compactly, define a vector sequence, c, by,

$$\begin{aligned} c(n) &= x(0) \text{ for } n = 0, \\ &= b(n-1) \text{ for } n > 0. \end{aligned} \tag{20}$$

Then (19) can be written,

$$x(n+1) = Ax(n) + c(n+1),$$

$$x(0) = c(0).$$

By induction, it is easy to verify that,

$$x(n) = \sum_{j=0}^{n} A^j\, c(n-j),$$

or, in the notation introduced in D.7,

$$x = (\lambda_n A^n) * c. \tag{21}$$

Since b is first-positive, it is easy to see from (20) that c is first-positive (If $x_\beta(0) > 0$, then $c_\beta(0) > 0$, so that c_β is certainly first-positive; if $x_\beta(0) = 0$, then the first non-zero element in the sequence c_β is the first non-zero element of b_β with non-zero index increased by 1, and must be positive since b is first-positive.) If A is Metzler, $\lambda_n A^n$ is first-positive by Theorem 3. Hence, by

Lemma 3, x is first-positive, and,

$$\nu(x)(\alpha) = \min_{\beta} [\ \nu(\lambda_n A^n)(\alpha,\beta) + \nu(c)(\beta)] \ . \tag{22}$$

Define, for any vector x,

D.22. $P(x) \triangleq \{\alpha \mid x_\alpha > 0\}$.

In (22), for each β, either $\beta \in P[x(0)]$ or $\beta \notin P[x(0)]$. The minimum can be taken separately over the two sub-sets and then the minimum of the two taken. Note that if $\beta \in P[x(0)]$, then, from (20), $\nu(c)(\beta) = 0$, while if $\beta \notin P[x(0)]$, then $\nu(c)(\beta) = \nu(b)(\beta) + 1$. Further, from Theorem 3, $\nu(\lambda_n A^n)(\alpha,\beta) = \delta(A)(\alpha,\beta)$. Substitution into (22) yields,

Theorem 5. If A is a Metzler matrix, b a first-positive sequence of vectors, $x(n+1) = Ax(n) + b(n)$, and $x(0) \geqq 0$, then x is a first-positive sequence, and

$$\nu(x)(\alpha) = \min \{ \min_{\beta\ \in P[x(0)]} \delta(A)(\alpha,\beta),\ 1 + \min_{\beta\ \notin P[x(0)]} [\delta(A)(\alpha,\beta) + \nu(b)(\beta)]\} \ .$$

Note that Theorem 5 implies that each component is positive before it can become negative. Further, a given component can be positive in two different ways. One is ultimately due to a positive initial component β which is linked to the given component α directly or indirectly. The other is through the emergence of a positive element in one component of the forcing term $b(n)$, which is then linked to the given component, α. The shortest of all these routes determines the length of time before the positive effect appears.

7. Genesis of a System of Differential Equations with Metzler Matrix as Jacobian

As a preliminary, we note, in the notation introduced in D.13 and D.22,

Lemma 9. If $A \geqq 0$ and $x \geqq 0$, then $P(Ax) = C(A)\ P(x)$.

Proof: $(A_\alpha) = \sum_\beta A_{\alpha\beta}\, x_\beta$. Since all terms are non-negative by assumption, $(Ax)_\alpha > 0$ if and only if, for some β, $A_{\alpha\beta} > 0$ and $x_\beta > 0$; but this holds if and only if, for some β, $\alpha C(A)\beta$ and $\beta \in P(x)$.

Now consider the system of differential equations,

$$\dot{x} = A\ \ + b(t), \tag{23}$$

where A is a Metzler matrix, $b(t) \geqq 0$ for all t, and $x(0) \geqq 0$. This clearly has the solution,

$$x(t) = e^{At}\, x(0) + \int_0^t e^{A(t-u)}\, b(u)\, du,$$

$$= y(t) + z(t), \tag{24}$$

where,

$$y(t) = e^{At}\, x(0), \tag{25}$$

$$z(t) = \int_0^t e^{A(t-u)}\, b(u)\, du. \tag{26}$$

By Theorem 4, $e^{At} \geqq 0$ for all $t > 0$, so that, from (25), $y(t) \geqq 0$ for all $t \geqq 0$. Also, $e^{A(t-u)} \geqq 0$ for $t > u$; since $b(u) \geqq 0$ for all u, by assumption,

$$e^{A(t-u)}\, b(u) \geqq 0,\ 0 \leqq u < t, \tag{27}$$

and therefore $z(t) \geq 0$ from (26). Combining these statements we see that,

$$x(t) \geq 0, \text{ and } P[x(t)] = P[y(t)] \cup P[z(t)], \text{ all } t > 0. \tag{28}$$

From Lemma 9, $P[y(t)] = C(e^{At})\, P[x(0)]$, all $t > 0$.

From Theorem 3, $C(e^{At}) = K^{\delta}(At)$ for $t > 0$; but since obviously $C(At) = C(A)$ for any matrix A and any scalar $t \neq 0$, it follows from the definition of $K^{\delta}(A)$, D.19, that $K^{\delta}(At) = K^{\delta}(A)$ for all $t > 0$.

$$P[y(t)] = K^{\delta}(A)\, P[x(0)] \text{ for } t > 0. \tag{29}$$

In particular, $P[y(t)]$ is independent of t for $t > 0$. Hence, as far as the effects of initial conditions go, any component that is going to be positive eventually is positive immediately. However, as will be seen below, the delay effects of the connectivity of the matrix affect the solution but in a different way.

If, for some component α, $(e^{A(t-u)}\, b(u))_{\alpha} = 0$ for all u, $0 \leq u < t$, then obviously, from (26), $(z(t))_{\alpha} = 0$. On the other hand, if b is assumed continuous, then if

$$(e^{A(t-u)}\, b(u))_{\alpha} > 0 \text{ for some } u,\ 0 \leq u < t,$$

it is positive in some interval and hence from (27) and (26), $(z(t))_{\alpha} > 0$. In symbols,

$$\begin{aligned} P[z(t)] &= \bigcup_{0 \leq u < t} P[e^{A(t-u)}\, b(u)] = \bigcup_{0 \leq u < t} C(e^{A(t-u)})\, P[b(u)] \\ &= \bigcup_{0 \leq u < t} K^{\delta}(A)\, P[b(u)] \\ &= K^{\delta}(A) \bigcup_{0 \leq u < t} P[b(u)], \text{ for } t > 0. \end{aligned} \tag{30}$$

The steps are the same as those leading to (29), together with a final step which uses (6).

The result can be made still more transparent with the aid of a nonzero index for functions of a continuous variable.

D.23. $\zeta(b)(\beta) = \inf\{t \mid t \geq 0,\ b_{\beta}(t) \neq 0\}$.

If $u < \zeta(b)(\beta)$, then $b_{\beta}(u) = 0$, and therefore $\beta \notin P[b(u)]$. Therefore, if $t \leq \zeta(b)(\beta)$, $\beta \notin P[b(u)]$ for all u, $0 \leq u < t$, and therefore,

if $t \leq \zeta(b)(\beta)$, $\beta \notin \bigcup_{0 \leq u < t} P[b(u)]$.

Suppose now $t > \zeta(b)(\beta)$. Then, by D.23, there exists u,

$\zeta(b)(\beta) \leq u < t$, $\beta \in P[b(u)]$,

and therefore,

$\beta \in \bigcup_{0 \leq u < t} P[b(u)]$.

Hence,

$\beta \in \bigcup_{0 \leq u < t} P[b(u)]$ if and only if

$$0 \leq \zeta(b)(\beta) < t. \tag{31}$$

For fixed b, $\zeta(b)$ is a function over a finite set. For any function, f, $f^{-1}(y)$ means the set $\{x \mid f(x) = y\}$; for any set S in the range of f, $f^{-1}(S)$ means the set $\{x \mid f(x) \in S\}$. In this notation,

$$0 \leq \zeta(b)(\beta) < t \quad \text{if and only if } \beta \in [\zeta(b)]^{-1} (<0,t)),$$

where $<0,t)$ is the interval closed on the left and open on the right.
Then,

$$\bigcup_{0 \leq u < t} P\, b(u) \quad = [\zeta(b)]^{-1} (<0, t)).$$

In combination with (30), (29), and (28), we can state,

Theorem 6.* Suppose A is a Metzler matrix, b continuous and non-negative, $\dot{x}(t) = Ax(t) + b(t)$ for all $t \geq 0$, and $x(0) \geq 0$. Then, $x(t) \geq 0$, all $t \geq 0$, and,

$$P[x(t)] = K^{\delta}(A)\, [P[x(0) \cup [\zeta(b)]^{-1}(<0,t)\ )] \quad \text{for } t > 0.$$

Notice again that the signs of all the components are completely determined by the sign patterns of the matrix of the initial conditions, A and of the forcing function b. At any time t, we find all components which are either positive at time 0 or have been nonzero at some time point before t; then take all components linked to them directly or indirectly by chains of nonzero entries in the matrix A. This set is precisely the set of positive components at time.

This theorem actually relates to more than the genesis of the dynamic system. The next result will study behavior at the starting-point, specifically, the qualitative behavior of the successive time derivatives of the different components of $x(t)$ at the point $t = 0$.

First, the non-negativity of a function and the first-positivity of the sequence of its derivatives are related.

Lemma 10. An infinitely differentiable vector function f is non-negative for $t \geq 0$ if and only if, for each $t \geq 0$, the sequence $\lambda_n f^{(n)}(t)$ is first-positive.

Proof: Suppose $f(t) \geq 0$, all $t \geq 0$, but for some $t_o \geq 0$, the sequence $\lambda_n f^{(n)}(t_o)$ is not first-positive. Then there exist β and n so that,

$f_\beta^{(r)}(t_o) = 0$ for $r < n$, $f_\beta^{(n)}(t_o) < 0$. But if $n = 0$, then $f_\beta(t_o) < 0$, contrary to hypothesis; if $n > 0$, then $f_\beta(t) < 0$ in some right-hand neighborhood of t_o, again contrary to hypothesis.

Conversely, suppose $\lambda_n f^{(n)}(t)$ is first-positive for all t. Then in particular, it is impossible that $f_\beta(t) < 0$ for any t and β, for then $f_\beta^{(o)}(t) < 0$, in which case the sequence $\lambda_n f^{(n)}(t)$ would not be first-positive.

Differentiate the system of differential equations (23) n times, and then set $t = 0$.

$$x^{(n+1)}(0) = Ax^{(n)}(0) + b^{(n)}(0). \tag{32}$$

By Lemma 10, the sequence $\lambda_n b^{(n)}(0)$ is first-positive; the matrix A is Metzler by assumption; and $x^{(0)}(0) = x(0) \geq 0$, by assumption. Hence, (32) constitutes a system of difference equations which satisfies all the hypotheses of Theorem 5.

Theorem 7. Under the hypotheses of Theorem 6, the sequence $\lambda_n x^{(n)}(0)$ is first positive, and

* The conclusion that $x(t) \geq 0$ already appeared in (1), Theorem *, p. 14.

$\nu(\lambda_n x^{(n)}(0))$

$$= \min\{ \min_{\beta \in P[x(0)]} \delta(A)(\alpha,\beta),\ 1 + \min_{\beta \notin P[x(0)]} \delta[(A)(\alpha,\beta) + \nu(\lambda_n b^{(n)}(0))]\}$$

Thus, a positive initial component causes every component indirectly connected to it to become positive in the right-hand neighborhood of the origin, but the order of growth (linear, quadratic, or whatever) depends on the length of the connecting chain through the matrix. Similarly, a forcing term will cause an order of growth in a component of x which is greater by 1 than the sum of the order of growth of the forcing term at zero and the length of the shortest chain to the x-component. These remarks are only valid for the first effect on the given component.

To illustrate, for an x-component which is initially zero, the growth is linear if either there is a chain of length 1 to a positive x-component or the forcing term for the given component is positive. The growth is quadratic if neither of these conditions hold and if one of the following three conditions is valid: (1) there is a chain of length 2 to a positive initial component; (2) there is a chain of length 1 to a component whose forcing term is increasing linearly from zero; (3) the forcing term for the given component is increasing quadratically.

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13. ABSTRACT

This paper studies the behavior in the neighborhood of the starting point of dynamic systems (solutions of difference or differential equations) whose Jacobians are Metzler matrices. (A Metzler matrix is one whose off-diagonal elements are non-negative.) A new concept, that of first-positivity of a sequence, is introduced; a sequence is first-positive if its first non-zero element is positive. First-positivity holds for a sequence of matrices or vectors if it holds for each component. It is shown that the sequence of powers of a Metzler matrix is first-positive; also, for each position in the matrix (defined by row and column), the number of steps to the first non-zero entry in the sequence is equal to the minimum length of a chain from the row to the column through non-zero entries in the Metzler matrix. From this, it is possible to express (a) the time of first-positivity of a specific component of the solution to a system of difference equations and (b) the order of increase of a specific component of the solution to a system of differential equations in terms of the connectivity properties of the governing matrix, the specification of the positive components of the starting values, and the first-positivity properties of the forcing function.

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